On Exponential Periodicity And Stability of Nonlinear Neural Networks With Variable Coefficients And Distributed Delays

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ABSTRACT

The exponential periodicity and stability of continuous nonlinear neural networks with variable coefficients and distributed delays are investigated via employing Young inequality technique and Lyapunov method. Some new sufficient conditions ensuring existence and uniqueness of periodic solution for a general class of neural systems are obtained. Without assuming the activation functions are to be bounded, differentiable or strictly increasing. Moreover, the symmetry of the connection matrix is not also necessary. Thus, we generalize and improve some previous works, and they are easy to check and apply in practice.

Keywords: Neural networks, exponential periodicity, distributed delays, Young inequality.

1. INTRODUCTION

In the past few years, the problem of dynamics of different classes of neural networks has been one of the most active areas of research and has attracted the attention of many researchers, we refer to [1-23]. The stability of neural networks with delays in the case of constant coefficients has been studied in Refs. [1-4,7,8,11-13,17-20]. However, most of them studied the dynamics of autonomous neural network model. For the non-autonomous neural network model with variable coefficients and distributed delays, up till now, the study works are very few. Moreover, studies on neural dynamical systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, bifurcation, and chaos. In many applications, the property of periodic oscillatory solutions and global exponential stability are of great interest. Recently, many authors [5,6,9,10,14,15,16,21,22,23] presented several new conditions for the exponential stability and periodic oscillatory solution of neural networks with delays in the case of constant coefficients. Cao [14] derived some simple sufficient conditions are given for global exponential stability and the existence of periodic solutions via the method of constructing suitable Lyapunov functionals. Song and Cao [15] presented some general sufficient conditions to ensure the global exponential stability and existence of periodic solutions of bi-directional associative memory (BAM) neural networks with delays and reaction-diffusion terms in terms of system parameters. In 2004, Li [16] imposed weaker conditions for the exponential stability than those reported by using the continuation theorem of coincidence degree theory and Lyapunov functions to study the existence and stability of positive periodic solutions for a class of BAM neural networks.

As is well known, the use of constant fixed delays in models of delayed feedback provides of a good approximation in simple circuits consisting of a small number of cells. However, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances, the signal propagation is not instantaneous and cannot be modelled with discrete delays and a more appropriate way is to incorporate continuously distributed delays. To the best of the our knowledge, few authors have considered the exponential periodicity and stability of nonlinear neural networks with delays [20-21]. However, the exponential periodicity and stability of nonlinear neural networks with variable coefficients and distributed delays have never been tackled.

The purpose of this paper is to study the exponential periodicity and stability of nonlinear neural networks with variable coefficients and distributed delays and give a set of criteria for the exponential periodicity and stability of the nonlinear neural networks by constructing new Lyapunov functionals and employing the Young inequality technique. In this paper, we do not require the activation functions to be bounded, differentiable or non-decreasing as are required in [4-7,11]; also we do not assume that the considered model has any equilibriums. We will see the obtained results improve and extend the main results on the exponential periodicity and stability for the neural networks given by researchers in [4-6,11].

2. SYSTEM DESCRIPTION AND PRELIMINARIES

In this paper, we deal with nonlinear continuous-time neural networks with variable coefficients and distributed delays described by the following functional differential equations

$$\dot{x}_i(t) = -f_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t))$$

$$+ \sum_{j=1}^{n} b_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-s) g_j(x_j(s)) ds + J_i(t), \quad (1)$$

where $i = 1, 2, \cdots, n$ and n denotes the number of neurons in a neural network; $x_i(t)$ corresponds to the state of the ith neuron at time t; $g_j(x_j(t))$ denote the activation functions of the jth neuron at time t; $a_{ij}(t)$, $b_{ij}(t)$ are the connection weights at the time t; $J_i(t)$ is the input periodic vector function with period ω, i.e. there exists a constant $\omega > 0$ such that $J_i(t+\omega) = J_i(t)(i = 1, 2, \cdots, n)$ for all $t \geq 0$. To obtain our results, we first give the following assumptions:
(H_1) $a_{ij}(t)$, $b_{ij}(t)$, $J_i(t)$, $i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$ are all continuous ω-periodic functions on R.
(H_2) $f_i : R \to R$ is differentiable and strictly monotone increasing, i.e. $d_i = \inf_{x \in R}\{f_i^{'}(x)\} > 0, i = 1, 2, \cdots, n$. For simplicity, let $\mathbf{D}$ be an $n \times n$ constant diagonal matrix with diagonal elements $d_i > 0$, $i = 1, 2, \cdots, n$.
(H_3) $g_i(\cdot)$ is globally Lipschitz continuous (GLC) and monotone increasing activation function; that is, for each $j \in \{1, 2, \cdots, n\}$, there exist L_j^* and L_j such that $0 < L_j^* \leq (g_j(x) - g_j(y))/(x-y) \leq L_j$ for all $x, y \in R$.
$(H_3^{'})$ $g_i(\cdot) \in GLC$ i.e. for each $j \in \{1, 2, \cdots, n\}$, $g_j : R \to R$ is globally Lipschitz continuous with Lipschitz constant $L_j > 0$, i.e. $|g_j(x) - g_j(y)| \leq L_j|x-y|$ for all $x, y \in R$.
The delay kernels $K(\cdot) = (K_{ij}(\cdot))_{n\times n}, i, j = 1, 2, \cdots, n$ are assumed to satisfy the following conditions simultaneously:
(i) $K_{ij} : [0, \infty) \to [0, \infty)$;
(ii) K_{ij} are bounded and continuous on $[0, \infty)$;
(iii) $\int_0^\infty K_{ij}(s) ds = 1$;
(iv) there exists a positive number ε such that $\int_0^\infty K_{ij}(s) e^{\varepsilon s} ds < \infty$.
The literature [24] has given some examples to meet the above conditions.
As a special case of neural system (1), the neural networks with constant input vector $J = (J_1, J_2, \cdots, J_n)^T \in R^n$ can reduce to the following functional differential equations:

$$\begin{aligned} \dot{x}_i(t) &= -f_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t)) + J_i \\ &+ \sum_{j=1}^{n} b_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-s) g_j(x_j(s)) ds \quad (2) \end{aligned}$$

Define $x_t(s) = x(t+s), s \in (-\infty, 0], t \geq 0$. Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, its norm is defined as

$$\|\phi_t\|_r = \sup_{-\infty \leq s \leq 0} |x(t+s)|_r,$$

where

$$|x(t+s)|_r = \Big[\sum_{i=1}^{n} |x_i(t+s)|^r\Big]^{1/r}.$$

Assume that the model (1) is supplemented with initial values of the type

$$x_i(t) = \phi_i(t), \ t \in (-\infty, 0]$$

in which $\phi_i(t), i = 1, 2, \cdots, n$ are continuous functions.
It is known obviously that model (2) serves as a general framework for neural network models. For instance, when $f_i(x_i(t)) = d_i x_i$, the neural network model becomes a delayed dynamical system studied in [23]; when $\mathbf{D}$ is an identity matrix, n is an even number and the weight matrices $A = (a_{ij})_{n\times n} = 0, B = (b_{ij})_{n\times n} = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix}$, and B_1, B_2 are $(n/2) \times (n/2)$ matrices, model (1) reduces to a BAM network with delay studied in [16].
Definition 1. System (1) is globally exponentially stable, if there are constants $\varepsilon > 0$ and $M \geq 1$ such that

$$\|x_t - y_t\|_r \leq M\|\phi - \varphi\|_r e^{-\varepsilon t} \quad (3)$$

for all $t > 0$; in which ϕ and φ are the initial functions of solutions $x(t)$ and $y(t)$, respectively.
Definition 2. The neural system (1) is said to be exponentially periodic if there exists one ω-periodic solution of the system and all other solutions of the system converge exponentially to it as $t \to +\infty$.
Definition 3 For any continuous function $V : R \to R$, Dini's time-derivative of $V(t)$ is defined as

$$D^+V(t) = \lim_{h\to 0^+} \sup \frac{V(t+h) - V(t)}{h}. \quad (4)$$

It is easy to see that if $V(t)$ is locally Lipschitz, then $|D^+V(t)| < \infty$.
Lemma 1 (Young's inequality [4]). Assume that $a > 0$, $b > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, then the following inequality:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad (5)$$

holds.

3. MAIN RESULTS

Let $C = C([-\infty, 0], R^n)$ be the Banach space of all continuous function from $[-\infty, 0]$ to R^n with the topology of uniform convergence. For any $\phi \in C$, let

$$\|\phi\| = \sup_{-\infty \leq t \leq 0} \Big(\sum_{i=1}^{n} |\phi_i(t)|^r\Big)^{1/r}.$$

Given any $\phi, \varphi \in C$, let

$$x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \cdots, x_n(t, \phi))^T,$$

$$x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \cdots, x_n(t, \varphi))^T$$

be the solutions of (1) starting from ϕ and φ, respectively.

Theorem 1. Suppose $(H_1)-(H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j>0$ and $a_{jj}(t)>0$, $j=1,2,\cdots,n$, such that

$$\begin{aligned}&-rd_i+ra_{ii}(t)L_i+(r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)|\\&+\sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r+(r-1)\sum_{j=1}^{n}|b_{ij}(t)|\\&+\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r<0,\end{aligned}\qquad(6)$$

where $r\geq 1$, then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

Proof. Define $x_t(\phi)=x(t+\theta,\phi),\theta\in(-\infty,0]$, then $x_t(\phi)\in C$ for all $t\geq 0$. Thus, it follows from (1) that

$$\begin{aligned}&\dot{x}_i(t,\phi)-\dot{x}_i(t,\varphi)\\=&-\Big(f_i(x_i(t,\phi))-f_i(x_i(t,\varphi))\Big)\\&+\sum_{j=1}^{n}a_{ij}(t)\Big(g_j(x_j(t,\phi))-g_j(x_j(t,\varphi))\Big)\\&+\sum_{j=1}^{n}b_{ij}(t)\int_{-\infty}^{t}K_{ij}(t-s)\\&\times\Big(g_j(x_j(s,\phi))-g_j(x_j(s,\varphi))\Big)ds\end{aligned}\qquad(7)$$

for $t\geq 0, i=1,2,\cdots,n$.
From $(H_1)-(H_3)$, it is easy to deduce that there exist d_i and $L_i\geq 0$ such that $|f_i(x_i(t,\phi))-f_i(x_i(t,\varphi))|\geq d_i|x_i(t,\phi)-x_i(t,\varphi)|$ and $|g_i(x_i(t,\phi))-g_i(x_i(t,\varphi))|\leq L_i|x_i(t,\phi)-x_i(t,\varphi)|$.
From (6), we can choose a small $\varepsilon>0$ such that

$$\begin{aligned}&\varepsilon-rd_i+ra_{ii}(t)L_i+(r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)|\\&+\sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r+(r-1)\sum_{j=1}^{n}|b_{ij}(t)|\\&+\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r<0, i=1,2,\cdots,n.\end{aligned}\qquad(8)$$

Consider the following positive definite Lyapunov functional candidate

$$\begin{aligned}V(t)=&\sum_{i=1}^{n}\lambda_i\Bigg[|x_i(t,\phi)-x_i(t,\varphi)|^re^{\varepsilon t}\\&+\sum_{j=1}^{n}\int_0^{\infty}K_{ij}(s)\\&\times\int_{t-s}^{t}|b_{ij}(s)|\Big|g_j(x_j(z,\phi))-g_j(x_j(z,\varphi))\Big|^r\\&\times e^{\varepsilon(z+s)}dzds\Bigg].\end{aligned}\qquad(9)$$

Computing the upper right derivative of $V(t)$ along the solution of (7) for $t\geq 0$, we get

$$\begin{aligned}&D^+V(t)\\=&\sum_{i=1}^{n}\lambda_i\Bigg[D^+\Big(|x_i(t,\phi)-x_i(t,\varphi)|^re^{\varepsilon t}\Big)\\&+\sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t,\phi))\\&-g_j(x_j(t,\varphi))\Big|^r\times e^{\varepsilon(t+s)}ds\\&-\sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t-s,\phi))\\&-g_j(x_j(t-s,\varphi))\Big|^r\times e^{\varepsilon t}ds\Bigg]\\\leq&e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i\Bigg[(\varepsilon-rd_i)|x_i(t,\phi)-x_i(t,\varphi)|^r\\&+ra_{ii}(t)L_i|x_i(t,\phi)-x_i(t,\varphi)|^r\\&+r\sum_{j=1,j\neq i}|a_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^{r-1}\\&\times\Big|g_j(x_j(t,\phi))-g_j(x_j(t,\varphi))\Big|\\&+r\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^{r-1}\\&\times\int_{-\infty}^{t}K_{ij}(t-s)\Big|g_j(x_j(s,\phi))\\&-g_j(x_j(s,\varphi))\Big|ds\\&+\sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t,\phi))\\&-g_j(x_j(t,\varphi))\Big|^r\times e^{\varepsilon s}ds\\&-\sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t-s,\phi))\\&-g_j(x_j(t-s,\varphi))\Big|^rds\Bigg].\end{aligned}\qquad(10)$$

By Lemma 1, it can follow that

$$\begin{aligned}&r\sum_{j=1,j\neq i}^{n}|a_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^{r-1}\\&\times\Big|g_j(x_j(t,\phi))-g_j(x_j(t,\varphi))\Big|\\\leq&(r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^r\\&+\sum_{j=1,j\neq i}^{n}|a_{ij}(t)|\Big|g_j(x_j(t,\phi))\\&-g_j(x_j(t,\varphi))\Big|^r,\end{aligned}\qquad(11)$$

and

$$r\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^{r-1}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{t} K_{ij}(t-s)\Big|g_j(x_j(s,\phi)) - g_j(x_j(s,\varphi))\Big|ds \\
\leq\ & (r-1)\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi)-x_i(t,\varphi)|^r \\
&+\sum_{j=1}^{n}|b_{ij}(t)|\int_{-\infty}^{t}K_{ij}(t-s)\Big|g_j(x_j(s,\phi)) \\
&-g_j(x_j(s,\varphi))\Big|^r ds. \qquad (12)
\end{aligned}
$$

Estimating the right of inequality (10) using (11)-(12) and the delay kernels' conditions (i)-(iv), we obtain

$$
\begin{aligned}
D^+V(t) \leq\ & e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i\Bigg[(\varepsilon - rd_i) + ra_{ii}(t)L_i \\
&+(r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| \\
&+\sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)| \\
&+(r-1)\sum_{j=1}^{n}|b_{ij}(t)| \\
&+\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r\Bigg] \\
&\times|x_i(t,\phi)-x_i(t,\varphi)|^r \leq 0. \qquad (13)
\end{aligned}
$$

Therefore

$$V(t) \leq V(0),\ t \geq 0. \qquad (14)$$

From (9), we derive

$$e^{\varepsilon t}\Big(\min_{1\leq i\leq n}\lambda_i\Big)\sum_{i=1}^{n}|x_i(t,\phi)-x_i(t,\varphi)|^r \leq V(t)$$

and

$$
\begin{aligned}
V(0) =\ & \sum_{i=1}^{n}\lambda_i\Bigg[|x_i(0,\phi)-x_i(0,\varphi)|^r \\
&+\sum_{j=1}^{n}|b_{ij}(0)|\int_{0}^{\infty}K_{ij}(s) \\
&\times\int_{-s}^{0}\Big|g_j(x_j(z,\phi)) - g_j(x_j(z,\varphi))\Big|^r \\
&\times e^{\varepsilon(z+s)}dzds\Bigg] \\
\leq\ & \max_{1\leq i\leq n}\lambda_i\Big[1 + L^r\sum_{j=1}^{n}|b_{ij}(0)| \\
&\times\int_{0}^{\infty}K_{ij}(s)\int_{-s}^{0}e^{\varepsilon(z+s)}dzds\Big]\|\phi-\varphi\|^r,
\end{aligned}
$$

where $L = \max_{1\leq i\leq n}\{L_i\}$ is a constant. Therefore, from (14), we get

$$\sum_{i=1}^{n}|x_i(t,\phi)-x_i(t,\varphi)| \leq M\|\phi-\varphi\|e^{-\frac{\varepsilon}{r}t},\ r\geq 1$$

for all $t \geq 0$, where

$$
\begin{aligned}
M =\ & \Big(\frac{1}{\min_{1\leq i\leq n}\lambda_i}\Big)^{1/r}\Bigg[\max_{1\leq i\leq n}\lambda_i\Big(1 + L^r\sum_{j=1}^{n}|b_{ij}(0)| \\
&\times\int_{0}^{\infty}K_{ij}(s)\int_{-s}^{0}e^{\varepsilon(z+s)}dzds\Big)\Bigg]^{1/r} \geq 1 \qquad (15)
\end{aligned}
$$

We can choose a positive integer N such that

$$Me^{-\frac{\varepsilon}{r}N\omega} \leq \frac{1}{3}.$$

Define a Poincaré mapping $P : C \to C$

$$\|P^N\phi - P^N\varphi\| \leq \frac{1}{3}\|\phi-\varphi\|. \qquad (16)$$

where $P^N\phi = x_{N\omega}(\phi)$. This implies that P^N is a contraction mapping. Therefore, there exists a unique fixed point $\phi^* \in C$ such that $P^N\phi^* = \phi^*$. So,

$$P^N(P\phi^*) = P(P^N\phi^*) = P\phi^*.$$

This shows that $P\phi^* \in C$ is also a fixed point of P^N, hence, $P\phi^* = \phi^*$, that is, $x_\omega(\phi^*) = \phi^*$. Let $x(t,\phi^*)$ be the solution of (1) through $(0,\phi^*)$. By using $J(t+\omega) = J(t)$ for $t \geq 0$, $x(t+\omega,\phi^*) = x(t,\phi^*)$ is also a solution of (1). Note that

$$x_{t+\omega}(\phi^*) = x_t(x_\omega(\phi^*)) = x_t(\phi^*) \text{ for } t \geq 0,$$

then

$$x(t+\omega,\phi^*) = x(t,\phi^*) \text{ for } t \geq 0,$$

This shows that $x(t,\phi^*)$ is a periodic solution of (1) with period ω. From (8), it is easy to see that all other solutions of (1) converge to this periodic solution exponentially as $t \to +\infty$.

Corollary 1. Suppose $(H_1)-(H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j > 0$ and $a_{jj}(t) > 0$, $j = 1, 2, \cdots, n$, such that

$$
\begin{aligned}
&-d_i + a_{ii}(t)L_i + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i \\
&+\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i < 0, \qquad (17)
\end{aligned}
$$

then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

Similar to the proof of Theorem 1, we can easily obtain the following results.

Theorem 2. Suppose $(H_1)-(H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j > 0$ and $a_{jj}(t) < 0$, $j = 1, 2, \cdots, n$, such that

$$
\begin{aligned}
&-rd_i + ra_{ii}(t)L_i^* + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| \\
&+\sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r + (r-1)\sum_{j=1}^{n}|b_{ij}(t)| \\
&+\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r < 0, \qquad (18)
\end{aligned}
$$

where $r \geq 1$, then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

Proof. The first step of proof is similar to Theorem 1. Then from (18), we can choose a small $\varepsilon > 0$ such that

$$\varepsilon - rd_i + ra_{ii}(t)L_i^* + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r + (r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r < 0, i = 1,2,\cdots,n. \quad (19)$$

Consider the following positive definite Lyapunov functional candidate

$$V(t) = \sum_{i=1}^{n}\lambda_i\Bigg[|x_i(t,\phi) - x_i(t,\varphi)|^r e^{\varepsilon t} + \sum_{j=1}^{n}\int_0^{\infty}K_{ij}(s) \times \int_{t-s}^{t}|b_{ij}(s)|\Big|g_j(x_j(z,\phi)) - g_j(x_j(z,\varphi))\Big|^r e^{\varepsilon(z+s)}dzds\Bigg]. \quad (20)$$

Since $a_{jj}(t) < 0$, we have

$$ra_{ii}(t)L_i|x_i(t,\phi) - x_i(t,\varphi)|^r \leq ra_{ii}(t)L_i^*|x_i(t,\phi) - x_i(t,\varphi)|^r. \quad (21)$$

Therefore, by using (21) and computing the upper right derivative of $V(t)$ along the solution of (20) for $t \geq 0$, we get

$$\begin{aligned}
&D^+V(t) \\
= &\sum_{i=1}^{n}\lambda_i\Bigg[D^+\Big(|x_i(t,\phi) - x_i(t,\varphi)|^r e^{\varepsilon t}\Big) + \sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t,\phi)) - g_j(x_j(t,\varphi))\Big|^r \times e^{\varepsilon(t+s)}ds \\
&- \sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t-s,\phi)) - g_j(x_j(t-s,\varphi))\Big|^r \times e^{\varepsilon t}ds\Bigg] \\
\leq &e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i\Bigg[(\varepsilon - rd_i)|x_i(t,\phi) - x_i(t,\varphi)|^r + ra_{ii}(t)L_i^*|x_i(t,\phi) - x_i(t,\varphi)|^r \\
&+ r\sum_{j=1,j\neq i}|a_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^{r-1} \times\Big|g_j(x_j(t,\phi)) - g_j(x_j(t,\varphi))\Big| \\
&+ r\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^{r-1} \times\int_{-\infty}^{t}K_{ij}(t-s)\Big|g_j(x_j(s,\phi)) - g_j(x_j(s,\varphi))\Big|ds \\
&+ \sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t,\phi)) - g_j(x_j(t,\varphi))\Big|^r \times e^{\varepsilon s}ds \\
&- \sum_{j=1}^{n}|b_{ij}(t)|\int_0^{\infty}K_{ij}(s)\Big|g_j(x_j(t-s,\phi)) - g_j(x_j(t-s,\varphi))\Big|^r ds\Bigg]. \quad (22)
\end{aligned}$$

By Lemma 1, it can follow that

$$\begin{aligned}
&r\sum_{j=1,j\neq i}^{n}|a_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^{r-1} \times\Big|g_j(x_j(t,\phi)) - g_j(x_j(t,\varphi))\Big| \\
\leq &(r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^r + \sum_{j=1,j\neq i}^{n}|a_{ij}(t)|\Big|g_j(x_j(t,\phi)) - g_j(x_j(t,\varphi))\Big|^r, \quad (23)
\end{aligned}$$

and

$$\begin{aligned}
&r\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^{r-1} \times\int_{-\infty}^{t}K_{ij}(t-s)\Big|g_j(x_j(s,\phi)) - g_j(x_j(s,\varphi))\Big|ds \\
\leq &(r-1)\sum_{j=1}^{n}|b_{ij}(t)||x_i(t,\phi) - x_i(t,\varphi)|^r + \sum_{j=1}^{n}|b_{ij}(t)|\int_{-\infty}^{t}K_{ij}(t-s)\Big|g_j(x_j(s,\phi)) - g_j(x_j(s,\varphi))\Big|^r ds. \quad (24)
\end{aligned}$$

Estimating the right of inequality (22) using (23)-(24) and the delay kernels' conditions (i)-(iv), we obtain

$$D^+V(t) \leq e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i\Bigg[(\varepsilon - rd_i) + ra_{ii}(t)L_i^* + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|$$

$$+(r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r\Big] \times|x_i(t,\phi)-x_i(t,\varphi)|^r \le 0. \quad (25)$$

Therefore

$$V(t)\le V(0),\ t\ge 0. \quad (26)$$

From (20), we derive

$$e^{\varepsilon t}\Big(\min_{1\le i\le n}\lambda_i\Big)\sum_{i=1}^{n}|x_i(t,\phi)-x_i(t,\varphi)|^r \le V(t)$$

and

$$\begin{aligned} V(0) &= \sum_{i=1}^{n}\lambda_i\Big[|x_i(0,\phi)-x_i(0,\varphi)|^r + \sum_{j=1}^{n}|b_{ij}(0)|\int_0^\infty K_{ij}(s) \\ &\quad \times\int_{-s}^{0}\Big|g_j(x_j(z,\phi))-g_j(x_j(z,\varphi))\Big|^r \times e^{\varepsilon(z+s)}dzds\Big] \\ &\le \max_{1\le i\le n}\lambda_i\Big[1+L^r\sum_{j=1}^{n}|b_{ij}(0)| \times\int_0^\infty K_{ij}(s)\int_{-s}^{0}e^{\varepsilon(z+s)}dzds\Big]\|\phi-\varphi\|^r, \end{aligned}$$

where $L=\max_{1\le i\le n}\{L_i\}$ is a constant. Therefore, we can get

$$\sum_{i=1}^{n}|x_i(t,\phi)-x_i(t,\varphi)|\le M\|\phi-\varphi\|e^{-\frac{\varepsilon}{r}t},\ r\ge 1$$

for all $t\ge 0$, where

$$M = \Big(\frac{1}{\min_{1\le i\le n}\lambda_i}\Big)^{1/r}\Big[\max_{1\le i\le n}\lambda_i\Big(1+L^r\sum_{j=1}^{n}|b_{ij}(0)| \times\int_0^\infty K_{ij}(s)\int_{-s}^{0}e^{\varepsilon(z+s)}dzds\Big)\Big]^{1/r}\ge 1 \quad (27)$$

We can choose a positive integer N such that

$$Me^{-\frac{\varepsilon}{r}N\omega}\le\frac{1}{3}.$$

Define a Poincaré mapping $P: C\to C$

$$\|P^N\phi-P^N\varphi\|\le\frac{1}{3}\|\phi-\varphi\|. \quad (28)$$

where $P^N\phi=x_{N\omega}(\phi)$. This implies that P^N is a contraction mapping. Therefore, there exists a unique fixed point $\phi^*\in C$ such that $P^N\phi^*=\phi^*$. So,

$$P^N(P\phi^*)=P(P^N\phi^*)=P\phi^*.$$

This shows that $P\phi^*\in C$ is also a fixed point of P^N, hence, $P\phi^*=\phi^*$, that is, $x_\omega(\phi^*)=\phi^*$. Let $x(t,\phi^*)$ be the solution of (1) through $(0,\phi^*)$. By using $J(t+\omega)=J(t)$ for $t\ge 0$, $x(t+\omega,\phi^*)=x(t,\phi^*)$ is also a solution of (1). Note that

$$x_{t+\omega}(\phi^*)=x_t(x_\omega(\phi^*))=x_t(\phi^*)\text{ for }t\ge 0,$$

then

$$x(t+\omega,\phi^*)=x(t,\phi^*)\text{ for }t\ge 0,$$

This shows that $x(t,\phi^*)$ is a periodic solution of (1) with period ω. From (19), it is easy to see that all other solutions of (1) converge to this periodic solution exponentially as $t\to+\infty$.

Corollary 2. Suppose $(H_1)-(H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j>0$ and $a_{jj}(t)<0$, $j=1,2,\cdots,n$, such that

$$-d_i+a_{ii}(t)L_i^*+\sum_{j=1,j\ne i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i<0, \quad (29)$$

then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

Theorem 3. Suppose that $(H_1)-(H_2)$ and $(H_3^{'})$ hold, and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j>0$, $j=1,2,\cdots,n$, such that

$$-rd_i+r|a_{ii}(t)|L_i+(r-1)\sum_{j=1,j\ne i}^{n}|a_{ij}(t)| + \sum_{j=1,j\ne i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r+(r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r<0, \quad (30)$$

where $r\ge 1$, then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

Corollary 3. Suppose that $(H_1)-(H_2)$ and $(H_3^{'})$ hold, and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j>0$, $j=1,2,\cdots,n$, such that

$$-d_i+|a_{ii}(t)|L_i+\sum_{j=1,j\ne i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i<0, \quad (31)$$

then the nonlinear neural networks (1) with variable coefficients and distributed delays is exponentially periodic.

When $J=(J_1,J_2,\cdots,J_n)$ is a constant vector, then for any constant $T\ge 0$ we have $J=J(t+T)=J(t)$ for $t\ge 0$. Thus, by the above results, when the sufficient conditions in Theorems 1-3 are satisfied, a unique periodic solution becomes a periodic solution with any positive constants as its period. period. So, the periodic solution reduced to a constant solution, that is, an equilibrium point. Furthermore, all other solutions globally exponentially converge to this equilibrium point as $t \to +\infty$. The unique equilibrium point of the delayed neural system (2) is globally exponentially stable. Suppose $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ are two solutions of model (2) with ϕ, φ as their initial functions. Then, by applying Theorems 1-3, we can easily obtain the following results, respectively.

Theorem 4. Suppose $(H_1) - (H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j > 0$ and $a_{jj}(t) > 0$, $j = 1, 2, \cdots, n$, such that

$$-rd_i + ra_{ii}(t)L_i + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r + (r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r < 0, \qquad (32)$$

where $r \geq 1$, then the nonlinear neural networks (2) with variable coefficients and distributed delays is exponentially stable.

Theorem 5. Suppose $(H_1) - (H_3)$ hold and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j > 0$ and $a_{jj}(t) < 0$, $j = 1, 2, \cdots, n$, such that

$$-rd_i + ra_{ii}(t)L_i^* + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r + (r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r < 0, \qquad (33)$$

where $r \geq 1$, then the nonlinear neural networks (2) with variable coefficients and distributed delays is exponentially stable.

Theorem 6. Suppose that $(H_1) - (H_2)$ and $(H_3^{'})$ hold, and the delay kernels $K(\cdot)$ satisfy (i)-(iv). If there exist constants $\lambda_j > 0$, $j = 1, 2, \cdots, n$, such that

$$-rd_i + r|a_{ii}(t)|L_i + (r-1)\sum_{j=1,j\neq i}^{n}|a_{ij}(t)| + \sum_{j=1,j\neq i}^{n}\frac{\lambda_j}{\lambda_i}|a_{ji}(t)|L_i^r + (r-1)\sum_{j=1}^{n}|b_{ij}(t)| + \sum_{j=1}^{n}\frac{\lambda_j}{\lambda_i}|b_{ji}(t)|L_i^r < 0, \qquad (34)$$

where $r \geq 1$, then the nonlinear neural networks (2) with variable coefficients and distributed delays is exponentially stable.

Remark 1. As consequence of Theorems 4-6, combining Corollaries 1-3, if choose $r = 1$ in Theorems 4-6, we can obtain a series corollaries of Theorems 4-6 for the exponential stability of the nonlinear neural networks (2) with variable coefficients and distributed delays.

Remark 2. It is obvious that the results obtained in this section improve and extend the results that were recently reported in [16,21]. In particular, we extend the main results of References [16] without assuming the boundedness of the activation functions. Therefore, this work gives some improvements to previous ones.

Remark 3. In Theorem 1, we give a new Lyapunov functional for the nonlinear neural network systems with distributed delays. This functional is constructed by improving the Lyapunov functional given by Li in [16].

4. A NUMERICAL EXAMPLE

Example Consider the following nonlinear neural networks with variable coefficients and distributed delays:

$$\begin{aligned}\dot{x}_i(t) &= -f_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)g_j(x_j(s))ds \\ &\quad + J_i(t), i = 1, 2 \end{aligned} \qquad (35)$$

associated with the the initial conditions: $x_i(s) = \phi_i(s)$ for $s \in [0, \infty)$, where $\phi_i(s)$ are bounded continuous functions on R. By taking $g_i(x) = \sin\left(\frac{1}{3}x\right) + \frac{1}{3}x$, so they are obviously Lipschitz continuous with Lipschitz constant $L_i = \frac{2}{3}, i = 1, 2$. And let's take $K_{ij}(t) = \frac{2}{\pi(1+t^2)}$, $i, j = 1, 2$, $f_1(x_1(t)) = 4(\sin(3x_1(t)) + 2)$, $f_2(x_2(t)) = 2\cos(x_2(t)) + 6$, $a_{ij}(t) = \frac{1}{5}\sin(i+j)t$, $b_{ij}(t) = \frac{1}{2}(\cos it - \sin jt)$ $(i, j = 1, 2)$; $J_i(t)$ are any continuous 2π-periodic functions for $i = 1, 2$. Then it is easy to verify that when $r = 1, \lambda_1 = \lambda_2 = 1$, (35) verifies all the sufficient condition of Theorem 1. Therefore, according to Theorem 1, (35) has a unique 2π-periodic solution and all solutions of (35) converge to the 2π-periodic solution.

5. CONCLUSIONS

In this paper, we have studied a class of continuous nonlinear neural networks with variable coefficients and distributed delays via employing Young inequality technique and Lyapunov method and established a series of new criteria on the exponential periodicity and stability for the model. we generalize and improve some previous works without assuming the activation functions are to be bounded, differentiable. These conditions are presented in terms of system parameters and have importance leading significance in designs and applications of nonlinear neural networks system with variable coefficients and distributed delays. An illustrative example is also worked out to demonstrate the effectiveness of our results. In addition, the methods given in this paper may be extended to study some more complex systems, such as nonlinear BAM neural networks with variable coefficients and distributed delays.

References

[1] J. D. Cao. A set of stability criteria for delayed cellular neural networks, IEEE Trans. Circuits Syst. I, vol. 48, pp. 494-498, 2001.

[2] J. D. Cao and D. M. Zhou. Stability analysis of delayed cellular neural networks, Neural Networks, vol. 11, no. 9, pp. 1601-1605, 1998.

[3] S. Arik and V. Tavsanoglu. Equilibrium analysis of delayed CNNs, IEEE Trans. Circuits Syst. I, vol. 45, pp. 168-171, 1998.

[4] X. Liao et al. Novel robust stability for interval-delayed Hopfield neural networks, IEEE Trans. Circuits Syst. I, vol. 48, pp. 1355-1359, 2001.

[5] J. D. Cao. Periodic oscillation and exponential stability of delayed CNNs, Phys. Lett. A, vol. 270, no. 3-4, pp. 157-163, 2000.

[6] S. Arik. Global robust stability of delayed neural networks, IEEE Trans. Circuits Syst. I, vol. 50, no. 1, pp. 156-160, 2003.

[7] S. Arik and V. Tavsanoglu. On the global asymptotic stability of delayed cellular neural networks, IEEE Trans. Circuits Syst. I, vol. 47, no. 4, pp. 571-574, 2000.

[8] X.Y. Lou, B.T. Cui, New criteria on global exponential stability of BAM neural networks with distributed delays and reaction-diffusion terms, International Journal of Neural Systems, vol. 17, no. 1, pp. 43-52, 2007.

[9] Z. G. Liu, A. P. Chen, J. D. Cao and L. H. Huang. Existence and global exponential stability of periodic solution for BAM neural networks with periodic coefficients and time-varying delays, IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 50, no. 9, pp. 1162-1173, 2003.

[10] J. D. Cao and L. Wang. Exponential stability and periodic oscillatory solution in BAM networks with delays, IEEE Transactions on Neural Networks, vol. 13, no. 2, pp. 457-463, 2002.

[11] J. D. Cao. Global stability conditions for delayed CNNS, IEEE Trans. Circuits Syst. I, vol. 48, pp. 1330-1333, 2001.

[12] S. Xu and J. Lam. A new approach to exponential stability analysis of neural networks with time-varying delays, Neural Networks, vol. 19, no. 1, pp. 76-83, 2006.

[13] X. Y. Lou, B. T. Cui. New LMI conditions for delay-dependent asymptotic stability of delayed Hopfield neural networks, Neurocomputing, vol. 69, no. 16-18, pp. 2374-2378, 2006.

[14] J. D. Cao. Global Exponential Stability and Periodic Solutions of Delayed Cellular Neural Networks, Journal of Computer and Systems Sciences, vol. 60, pp. 38-46, 2000.

[15] Q. K. Song and J. D. Cao. Global exponential stability and existence of periodic solutions in BAM networks with delays and reaction-diffusion terms, Chaos, Solitons and Fractals, vol. 23, pp. 421-430, 2005.

[16] Y. K. Li. Existence and stability of periodic solution for BAM neural networks with distributed delays, Applied Mathematics and Computation, vol. 159, pp. 847-862, 2004.

[17] K. Matsuoka. Stability conditions for nonlinear continuous neural networks with asymmetric connection weights, Neural Networks, vol. 5, no. 3, pp. 495-500, 1992.

[18] X. Y. Lou and B. T. Cui. Global asymptotic stability of delay BAM neural networks with impulses, Chaos, Solitons and Fractals, vol. 29, no. 4, pp. 1023-1031, 2006.

[19] X. Y. Lou and B. T. Cui. Absolute exponential stability analysis of delayed bi-directional associative memory neural networks, Chaos, Solitons and Fractals, vol. 31, no. 3, pp. 695-701, 2007.

[20] H. T. Lu. On stability of nonlinear continuous-time neural networks with delays, Neural Networks, vol. 13, no. 10, pp. 1135-1143, 2000.

[21] C. Y. Sun and C. B. Feng. Exponential periodicity and stability of delayed neural networks, Mathematics and Computers in Simulation, vol. 66, pp. 469-478, 2004.

[22] H. Huang, D. W. C. Ho and J. D. Cao. Analysis of global exponential stability and periodic solutions of neural networks with time-varying delays, Neural Networks, vol. 18, no. 2, pp. 161-170, 2005.

[23] K. Gopalsamy and X. Z. He. Stability in asymmetric Hopfield nets with transmission delays, Physica D, vol. 76, pp. 344-358, 1994.

[24] S. Mohamad and K. Gopalsamy. Dynamics of a class of discrete-time neural networks and their continuous-time counterparts, Math. Comput. Simulation, vol. 53, no. 1-2, pp. 1-39, 2000.

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