Two Fuzziness Indexes Proposed by Kaufmann: observations about them

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Abstract— **Professor Arnold Kaufmann did propose at least two types of indexes for estimating fuzziness in finite standard fuzzy sets. First one has an analogue formulation to that stated by Claude Shannon for measuring uncertainty in a given system. Shannon formulation estimates one type of uncertainty classified as conflict. The present paper will reveal the inconvenience of such an index for measuring fuzziness phenomena. In addition, it is proved algebraic equivalence between another index posed by Kaufmann and a fuzziness index proposed by Ronald Yager.**

Index Terms— **Entropy, Fuzziness, Fuzzy sets, Uncertainty Indexes, Uncertainty Measure.**

1. INTRODUCTION

THIS paper presents several reasons for low applicability of a fuzziness index proposed by A. Kaufmann. Fuzziness is an unavoidable phenomenon when uncertainty is being evaluated during fuzzy modeling. In spite of similarities between Kaufmann index and others commonly used indexes, the index has problems when it is used for estimating fuzziness in finite standard sets. In this way, any fuzziness measurement will yield unreliable value in respecting the measured property. In this work, evidences about non-desirable properties of Kaufmann index are given comparing its behavior with other fuzziness indexes. In what follows the Kaufmann Index just cited will be labeled Kaufmann´s Entropy Index. In addition, this paper is devoted to prove the equality of algebraic forms (under certain mathematical considerations) of Kaufmann´s Linear Fuzziness Index and Yager´s Normalized Fuzziness Index.

The paper has the following structure. In section 2, the Shannon entropy formulation is shown. Section 3 describes the fuzziness index proposed by Kaufmann as an adaptation of Shannon's Entropy Index. Section 4 proves the unsuitable use of Kaufmann's Entropy Index for measuring the fuzziness. Section 5 establishes the equality of algebraic forms of Kaufmann´s Linear Fuzziness Index and Yager´s Normalized Fuzziness Index.

2. A DESCRIPTION OF SHANNON ENTROPY FORMULATION

Assume that a system can be in any state from a given set of *N* possible states, $X = \{x_1, x_2, ..., x_N\}$, and consider that a given probability distribution was stated for the set *X*, $\mathbf{p} = \langle p_1, p_2, ..., p_N \rangle$, with p_i the probability measure for the x_i state. The uncertainty grade, $S(\mathbf{p})$, of the mentioned system, can be calculated using:

$$S(\mathbf{p}) = -\sum_{i=1}^{N} p_i \log_b p_i \quad (1)$$

The value obtained in Eq. (1) is called "Entropy" by Shannon in [1], following its similarity with the mathematical formulation for quantifying entropy phenomenon in physical systems. Two significant properties can be appreciated from Shannon Entropy:

-- *Any system with only one possible state cannot have Shannon Entropy*. The fact of having only one possible state x_i in the state space of a system, can be modeled using a probability distribution $\mathbf{p} = \langle p_1, p_2, ..., p_N \rangle$, $p_i = 1$ and $p_j = 0, \forall j \neq i$, $i, j \in \mathbb{N}_N$, $\mathbb{N}_N = \{1, ..., N\}$. In this situation, the Shannon Entropy value is $S(\mathbf{p}) = 0$.

-- *A system has maximum Shannon Entropy when all possible system states have equal probability of being reached*. In this case, the probability distribution is **p**, $p_i = 1/N, \forall i$. Therefore, a system with uniform probability distribution has maximum Entropy value in the Shannon sense, and $S(\mathbf{p}) = \log_b N$.

In agreement with the two previously presented properties, for any system, Shannon Entropy varies between two known limits: $0 \leq S(\mathbf{p}) \leq \log_b N$.

3. KAUFMANN FUZZINESS INDEX AS AN ADAPTATION OF SHANNON ENTROPY

Professor A. Kaufmann modifies Shannon formulation in order to obtain a fuzziness index. Such an index varies in the $[0,1]$ interval and is calculated as:

$$\breve{S}(\mathbf{p}) = -\frac{1}{\ln N} \sum_{i=1}^{N} p_i \ln p_i \quad (2)$$

In an example described in [2], page 26, Kaufmann proposes an analogy between his formulation and

Shannon´s measure:

Let us suppose that the discourse universe $X = \{x_1, x_2, ..., x_N\}$ is available and a fuzzy set $F = \sum_{i=1}^{N} \mu_F(x_i)/x_i$ is determined over *X*. Considering that $\mu_F(x_i)$ is the membership grade of x_i to *F* and $\sum$ represents the union operation over fuzzy sets $\mu_F(x_i)/x_i$, Kaufmann estimates fuzziness of *F* as:

$$\breve{S}(\boldsymbol{\pi}_F) = -\frac{1}{\ln N}\sum_{i=1}^{N} \pi_i \ln \pi_i \tag{3}$$

with $\boldsymbol{\pi}_F = \langle \pi_1, \pi_2, ..., \pi_N \rangle$ and

$$\pi_i = \pi(x_i) = \frac{\mu_F(x_i)}{\sum_{i=1}^{N}\mu_F(x_i)}. \tag{4}$$

4. Observations about Kaufmann Formulation

Any fuzziness measurement function *f*, is defined as $f : P(X) \rightarrow C$, with $P(X)$ the power set or family of standard fuzzy subsets of *X* and $C = \Re^+ \cup \{0\}$. Three axiomatic principles, or axioms for abbreviating, of any fuzziness measurement function, *f*, are explained in [6]. Those axioms are:

-- *Axiom 1*. $f(A) = 0$ if, and only if, *A* is a crisp set (non-fuzzy set).

-- *Axiom 2*. If $A \prec B$, then $f(A) \le f(B)$, where $A \prec B$ indicates that *B* is more fuzzy than *A*. However, such a requirement is dependent of the definition of "*more fuzzy than*".

-- *Axiom 3*. Function $f(A)$ assumes the maximum value if, and only if, *A* is *maximally fuzzy*.

Kaufmann's formulation for estimating fuzziness, given in (3) and (4), does not fulfill these axioms as it will be shown through the *observations 1 - 4*.

Observation 1

Let F any fuzzy set such that $\mu_F(x_i) = c$, $\forall x_i \in X$, $i = 1,..., N$. According to Eq. (3) and Eq. (4):

$$\begin{aligned}\breve{S}(\boldsymbol{\pi}_F) &= -\frac{1}{\ln N}\sum_{i=1}^{N} \frac{\mu_F(x_i)}{\sum_{i=1}^{N}\mu_F(x_i)} \ln \frac{\mu_F(x_i)}{\sum_{i=1}^{N}\mu_F(x_i)} \\ &= -\frac{1}{\ln N} N\left(\frac{c}{cN}\ln\frac{c}{cN}\right) \\ &= -\frac{1}{\ln N}(\ln 1 - \ln N) \\ &= 1\end{aligned} \tag{5}$$

An effect of (5) is, that *degrees of fuzziness* of sets with homogeneous membership functions is not distinguished. This consequence of Eq. (5) disagrees with *Axiom 2* since that as $c \rightarrow 0$ or $c \rightarrow 1$ sets are less fuzzy, and its corresponding values $\breve{S}(\cdot)$ would be smaller ones.

Observation 2

Another consequence of Eq. (5) is that the criterion of maximal fuzziness could be *to display an uniform membership function*, this criterion is not generally correct one.

Observation 3

Kaufmann formulation assigns fuzziness value different of zero to crisp sets.

Every crisp set F with q elements, $q < N$, has a fuzziness value:

$$\begin{aligned}\breve{S}(\boldsymbol{\pi}_F) &= -\frac{1}{\ln N}\sum_{i=1}^{q}\frac{1}{q}\ln\frac{1}{q} \\ &= -\frac{1}{\ln N} q\left(\frac{1}{q}\ln\frac{1}{q}\right) \\ &= -\frac{1}{\ln N}(\ln 1 - \ln q) \\ &= \frac{\ln q}{\ln N}\end{aligned} \tag{6}$$

The last expression in Eq. (6) implies that every crisp set has a value of fuzziness different to zero. In this way the *Axiom 1* is not fulfilled.

A notorious instance of preceding situation is the discourse set X. This set can not exhibit fuzziness degree different of zero because it is a crisp set. The set $X \in P(X)$ is a fuzzy set with a homogeneous membership function where $\mu_X(x_i) = 1$, $\forall x_i \in X$, $i = 1,..., N$. Kaufmann's Entropy Index would yield, according to Eq. (6), a fuzziness value different to zero for the set *X* and therefore *Axiom 1* is violated. In accordance with Eq. (5) Kaufmann's Entropy Index estimates a maximal fuzziness value to crisp set *X*, this contradicts *Axiom 3* because a crisp set can not be maximally fuzzy set. These results are recognized by Kaufmann as a wrong outcome of his index. However, he explains it as a consequence of using relative membership grades as it appears in Eq. (4).

Observation 4

Let *F* and *G* be two crisp sets. Suppose that cardinality of *F*, $|F|$, is *m* and $|G| = p$, with $m, p \neq 0$. Using (3):

$$\breve{S}(\boldsymbol{\pi}_F) = \frac{\ln m}{\ln N} \neq 0 \text{ and } \breve{S}(\boldsymbol{\pi}_G) = \frac{\ln p}{\ln N} \neq 0$$

In addition, if $m < p$, $\breve{S}(\boldsymbol{\pi}_F) < \breve{S}(\boldsymbol{\pi}_G)$.

The previous result means that two *crisp* sets can be compared according to its *fuzziness*. This result implies that the *Axiom 1* is not fulfilled.

Observation 5

Finally, the doubtful applicability of Kaufmann's Entropy Index for fuzziness measurement is evident when it is compared with others proposed indexes, such as: Linear Index and Quadratic Index, both of them formulated in [2], and De Luca-Termini Index [7]-[9]. A conception of fuzziness property establishes that *as difference between a fuzzy set and its nearest crisp set is smaller, is less fuzzy*. This fuzziness conception implies that a fuzzy set *A* is less fuzzy than another fuzzy set *B* if

$$C = \{x | \mu_A(x) \ge \mu_B(x), \mu_A(x) > 0.5, \mu_B(x) > 0.5\},$$

$$D = \{x | \mu_A(x) \le \mu_B(x), \mu_A(x) \le 0.5, \mu_B(x) \le 0.5\},$$

$C \cap D = \phi$ and $C \cup D = X$.

The behavior of all these indexes can be shown by means of two cases.

Case 1. Let $X = \{x_1, x_2, x_3, x_4\}$ a discourse universe and $F = \{1/x_1, 0.9/x_2, 0.9/x_3, 0.8/x_4\}$ and $G = \{0.9/x_1, 0.8/x_2, 0.9/x_3, 0.7/x_4\}$, two fuzzy sets on *X*. It is evident that $\mu_F(x) \geq \mu_G(x) \quad \forall x \in X$, and $\mu_F(x) \geq 0.5$, $\mu_G(x) \geq 0.5$, it means that set *F* is less fuzzy than set *G*. However, as it is shown in Table I., Kaufmann's Entropy Index establishes the opposite statement.

TABLE I
FUZZINESS IN SETS *F* AND *G*, USING LINEAR, QUADRATIC, DE-LUCA TERMINI, AND KAUFMANN INDEXES.

	Linear	Quadratic	De Luca-Termini (scaled)	Kaufmann (scaled)
F	0.2000	0.2449	0.4150	0.9978
G	0.3500	0.3873	0.6353	0.9963

Case 2. Let $X = \{x_1, x_2, x_3, x_4\}$ a discourse universe and $H = \{0.2/x_1, 0.3/x_2, 0.1/x_3, 0.2/x_4\}$ and $I = \{0.4/x_1, 0.3/x_2, 0.1/x_3, 0.4/x_4\}$, two fuzzy sets on *X*. It is clear that $\mu_H(x) \leq \mu_I(x) \quad \forall x \in X$, and $\mu_H(x) \leq 0.5$, $\mu_I(x) \leq 0.5$. It implies that set *H* is less fuzzy than set *I*. Kaufmann's Entropy Index declares a differing assertion, as it can be observed in Table II:

TABLE II
FUZZINESS IN SETS *H* AND *I*, USING LINEAR, QUADRATIC, DE-LUCA TERMINI, AND KAUFMANN INDEXES.

	Linear	Quadratic	De Luca-Termini (scaled)	Kaufmann (scaled)
H	0.4000	0.4243	0.6985	0.9528
I	0.6000	0.6481	0.8230	0.9277

5. ABOUT ALGEBRAIC EQUALITY BETWEEN LINEAR FUZZINESS AND NORMALIZED YAGER INDEXES

A feature of indexes for calculating set fuzziness is its complete correspondence with one of the two groups of mathematical functions for measuring this property. The first group requires fixing the difference between the set and its nearest crisp set. The second one needs to calculate that difference with respect to its complement set. It can be proved that the Linear Fuzziness Index formulated by Kaufmann [2] (different to Kauffman's Entropy Index described in previous sections) and fuzziness index proposed by Yager [10], have the same algebraic form, even though they belong to different groups. The preceding assertion will be proved in the following proposition.

Proposition. If the Hamming distance and the standard definition of fuzzy set complement are used, *Linear fuzziness Index* proposed by Kaufmann and *Normalized Fuzziness Index* formulated by Yager, both of them acquire the same algebraic form, and as a consequence, generate the same value.

Proof. When fuzziness in a fuzzy set *A* is estimated using the Linear Fuzziness Index, the membership values of elements of *X* to *nearest crisp set of A*, C_A, must be obtained. The membership values to the set C_A, is fixed as:

$$\mu_{C_A}(x) = \begin{cases} 1, & \mu_A(x) > 0.5 \\ 0, & \mu_A(x) \leq 0.5 \end{cases} \tag{7}$$

Let $A_>$ and $A_\leq$ be crisp sets stated in the following way: $A_> = \{x | \mu_{C_A}(x) = 1\}$ and $A_\leq = \{x | \mu_{C_A}(x) = 0\}$. These two crisp sets satisfy the following properties: $A_> \cap A_\leq = \phi$ and $A_> \cup A_\leq = X$, it means that $\{A_>, A_\leq\}$ is a *partition* of *X* and as a consequence $|A_>| + |A_\leq| = |X|$.

Kaufmann´s Linear Fuzziness Index, in case of using Hamming distance, $D_H(\cdot,\cdot)$, calculates fuzziness by means of the mathematical expression:

$$\hat{v}(A) = \frac{2}{|X|} D_H(A, C_A) \tag{8}$$

$$\hat{v}(A) = 2\frac{\sum_{x \in X} |\mu_A(x) - \mu_{C_A}(x)|}{|X|}$$

Considering that the family of sets $\{A_>, A_\leq\}$ is a partition of *X*:

$$\hat{v}(A) = 2\frac{\sum_{x \in A_>} (1 - \mu_A(x)) + \sum_{x \in A_\leq} \mu_A(x)}{|X|}$$

Finally, Eq. (9) shows the mathematical expression for $\hat{v}$(A):

$$\hat{v}(A) = 2\frac{|A_>| - \sum_{x \in A_>} \mu_A(x) + \sum_{x \in A_\leq} \mu_A(x)}{|X|} \tag{9}$$

Whenever fuzziness of a fuzzy set *A* is obtained by means of the Yager´s Normalized Fuzziness Index, it is necessary to determine the difference with respect to its complement set $\overline{A}$. Yager´s Normalized Fuzziness Index is:

$$\hat{f}_Y(A) = 1 - \frac{D(A, \overline{A})}{|X|} \tag{10}$$

$D(A, \overline{A})$: is a metric difference between *A* and $\overline{A}$.

If $D(\cdot,\cdot)$ is obtained through Hamming distance, $D_H(\cdot,\cdot)$, Yager´s Normalized Fuzziness Index assumes the following form:

$$\hat{f}_Y(A) = 1 - \frac{\sum_{x \in X} |\mu_A(x) - \mu_{\overline{A}}(x)|}{|X|} \tag{11}$$

If the standard fuzzy complement operation for establishing $\overline{A}$ is used, (11) takes the form:

$$\hat{f}_Y(A) = 1 - \frac{\sum_{x \in X} |2\mu_A(x) - 1|}{|X|} \tag{12}$$

The following algebraic manipulations on (12) can be done:

$$= 1 - \frac{\sum_{x \in A_>} (2\mu_A(x) - 1) + \sum_{x \in A_\leq} (1 - 2\mu_A(x))}{|X|}$$

$$= 1 - \frac{2\sum_{x \in A_>} \mu_A(x) - |A_>| + |A_\leq| - 2\sum_{x \in A_\leq} \mu_A(x)}{|X|}$$

$$= 1 - \frac{2\sum_{x \in A_>} \mu_A(x) - 2\sum_{x \in A_\leq} \mu_A(x) - |A_>| + |A_\leq|}{|X|}$$

$$= 1 - \frac{2\sum_{x\in A_>} \mu_A(x) - 2\sum_{x\in A_\le} \mu_A(x) - |A_>| + |X| - |A_>|}{|X|}$$

$$= 1 - \frac{2\sum_{x\in A_>} \mu_A(x) - 2\sum_{x\in A_\le} \mu_A(x) - 2|A_>| + |X|}{|X|}$$

$$= \frac{|X| - 2\left(\sum_{x\in A_>} \mu_A(x) - \sum_{x\in A_\le} \mu_A(x)\right) - |X| + 2|A_>|}{|X|}$$

The final expression for $\hat{f}_Y(A)$ is

$$\hat{f}_Y(A) = 2\frac{|A_>| - \sum_{x\in A_>} \mu_A(x) + \sum_{x\in A_\le} \mu_A(x)}{|X|} \qquad (13)$$

By comparing Eq. (9) and Eq. (13), it is obvious that whenever Hamming distance is used:

$$\hat{\nu}(A) = \hat{f}_Y(A) \qquad (14)$$

6. Conclusions

A fuzziness index proposed by Professor A. Kaufmann was analyzed in order to detect its capability for measuring such a phenomenon. Major problems were evident when using Kaufmann's Entropy Index to measure fuzziness. Some of those problems about Kaufmann's Entropy Index were found comparing its behavior with other fuzziness indexes. Other difficulties of Kaufmann's Entropy Index are due to non-adhere the axioms defined for that type of indexes. In this way, other indexes for fuzziness measurement were presented in order to list available more effective indexes. The reader has the option of use one of those indexes looking for better fuzziness estimation.

The Kaufmann's Entropy Index is important because it was one of the first indexes proposed for measuring fuzziness. Moreover, Kaufmann as author of [2], is an important collaborator to formalize and to concrete some notions on fuzzy set theory. Finally, Kaufmann's entropy index has been cited in some literary sources: [3] – [5].

It was proved that *Linear fuzziness Index* proposed by Kaufmann and *Normalized Fuzziness Index* formulated by Yager have the same algebraic form, even though they belong to different groups. This statement implies that these two indexes produce numerically identical results, and therefore, it blurs the supposed *clear distinction* between the two categories of fuzziness indexes mentioned in section 5.

7. References

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Received: Nov. 2008. Accepted: Feb. 2009.