- ORIGINAL ARTICLE -

Haar-Like Wavelets over Tetrahedra

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Abstract

In this paper we define a Haar-like wavelets basis that form a basis for $L^2(T,S,\mu)$, μ being the Lebesgue measure and S the σ-algebra of all tetrahedra generated from a subdivision method of the T tetrahedron. As $3D$ objects are, in general, modeled by tetrahedral grids, this basis allows the multiresolution representation of scalar functions defined on polyhedral volumes, like colour, brightness, density and other properties of an $3D$ object.

Keywords: subdivision methods, volumetric data, multiresolution analysis, tetrahedral meshes.

1 Introduction

Wavelets have been appearing in many pure and applied areas of science and engineering as a versatile tool for representing general functions or big data sets. Up to 1994 all the results were referred to the classic wavelet theory, i.e all the basis functions are dilations and translations of a fixed function called *mother wavelet*. This theory is known as first generation wavelets. The traditional works on wavelets were done by Daubechies [1], Mallat [2], Chui [3]. The Haar wavelet, Daubechies wavelets, B-wavelets associated to B-splines are examples of first generation wavelets.
First generation wavelets have a disadvantage: they can only represent equally spaced data. Although signalsand stadistic data are so distributed, several other data are irregularly distributed and this motivated a first extension in the wavelet theory. At the beginning of the ninetieth-decade the *second generation wavelets* appeared. Within this new framework many researchers defined wavelets defined over triangles because they are the basic building block constructors of surfaces. Having wavelets over triangles allowed to define wavelets over polygonal surfaces.
The most general techniques for building second generation wavelets are the *lifting scheme* and the *surfaces subdivision*.
The surfaces subdivision is a technique developed by Lounsbery [4] who extended the wavelet theory to arbitrary topological surfaces. Afterwards, and inspired in [4], Schroeder and Sweldens [5] used subdivision and lifting to provide an efficient methodology for costum-design construction of wavelets. They focused their work on wavelets representation of functions defined on a sphere. They built the so called spherical wavelets, which are wavelets defined over spherical triangles, and used them to represent some functions defined on the sphere: topographic data, bidirectional reflection distribution functions and illumination.
Nielson [6] defined piecewise constant wavelets over nested triangulated domains and applied them to the problem of multiresolution analysis of flow over a spherical domain.
The above cited works addressed the problem of defining wavelets bases over surfaces and so a multiresolution representation of surfaces [4] and a multiresolution representation of funcions over surfaces is possible [5].
Following the successful steps for defining wavelets on surfaces, in order to define wavelets on a volume it is neces sary to have a representation of the volume using simpler block constructors. Different alternatives have been proposed for modeling volumes; tetrahdedra are preferred over hexahedra because they are better for modeling objects nearby frontiers. But in this work we don't focus our attention in this problem. In fact we suppose we have already got a tetrahedrilized of the volume and we want to represent a function defined over it. Many applications such as those involving Internet $3D$ models, collaborative CAD, interactive visualization and multi-player video games cope with the problem of representing

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functions over a volume. However and up to our knowledge, not many articles can be found in the literature that give a concrete wavelet construction for representing functions defined on volumes modeled by tetrahedral meshes. Similar bases but on $2D$ triangles can be seen in [7] and [8].

Why is important to have wavelets defined over tetrahedra? Because tetrahedra are used for modeling objects. Having a wavelets basis over tetrahedra will allow to define wavelets basis over volumes. Taking into account the Haar like wavelets defined in [9], in this paper we give a much simpler expression for them that allows an easier numerical implementation. The wavelets form a basis for $L^2(T,S,\mu)$, μ being the Lebesgue measure and S the σ-algebra of all tetrahedra generated from a subdivision method of the T tetrahedron that enables the multiresolution representation of functions defined over a tetrahedron T and, consequently, the multiresolution representation of functions defined over a tetrahedrilized volume. Also, we include a table where we show the initial values and the values obtained at coarser resolutions. In this way the interested reader can verify the values we have obtained.

The rest of the paper is organized as follows. Section 2 describes Bey's method for subdividing a tetrahedron. Section 3 defines the Haar like wavelets and the multiresolution analysis spanned by them. Section 4 shows an application example in which density function associated to the interior and surface of a given tetrahedron is represented using that wavelets. Finally, Section 5 points out conclusions and future work.

2 Proposed approach

The problem we focus on, is the definition of Haar like wavelets which form a basis for $L^2(T,S,\mu)$, μ being the Lebesgue measure and S the σ-algebra of all tetrahedra generated from Bey's subdivision method of a given tetrahedron T. Our goal is to give a multiresolution representation of piecewise constant functions defined over a tetrahedral regular mesh using this basis. This basis is inspired by the construction of orthogonal Haar wavelets for general measures, (see [10]) where it is shown that the Haar wavelets form an unconditional basis.

2.1 Tetrahedral meshes

Different alternatives have been proposed for modeling volumes [6, 11, 12, 13], however most of them may be classified in two different types:

- the ones that represent the volume through its surface or union of surfaces and decompose it on simpler constructors like triangles.
- the ones that represent the volume by simpler volume constructors such as cubes or tetrahedra, (analog to triangles for surfaces). Tetrahedra have an advantage over cubes: they model better $3D$ objects because they can adapt better to the frontier.

Given a $3D$ object, there are several methods for getting a tetrahedrilized of it. We don't focus on this. We suppose we are given the tetrahedrilized object on which the scalar field is defined. Also we suppose these data are regularly distributed. Tetrahedral meshes which model such data are called *regular* or *nested* meshes because they allow to define nested spaces such as those needed in wavelet theory.

In order to define the wavelets, it is necessary to adopt a subdivision method for a tetrahedron. This will allow to define the nested spaces needed in the framework of multiresolution analysis. We choose Bey's subdivision method for a tetrahedron, [14].

2.2 Bey's method

Definition: Two tetrahedra T_1 and T_2 are said to be congruent, if they can be made to coincide by a rigid motion and a positive or negative scaling, that is to say, if there exists a scaling factor $c \neq 0$, a traslation vector x and an orthogonal matrix Q such that:

$$T_1 = x + cQT_2 := \{x + cQx' : x' \in T_2\}.$$

For subdividing a given tetrahedron T, which is called *father*, we first connect the midpoints of the edges of each face triangle of T. Then we cut the four tetrahedra at the corners which are congruent with T. In the interior of the remaining octahedron there are three parallelograms as we can see in Figure 1. Cutting the octahedron along two of this parallelograms, we obtain other four subtetrahedra. Each choice of two parallelograms corresponds to one of three possible diagonals, as shown in Figure 1. The eight subtetrahedra, called sons (Figure 2), have equal volume but the interior ones are generally not congruent with T.

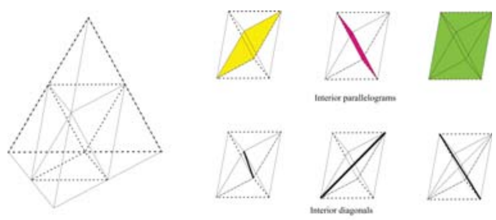

Figure 1: Bey's subdivision method for a tetrahedron. Interior Octahedron.

3 Haar like wavelets over a tetrahedron

Now we give an order for the eight obtained subtetrahedra in the following way (see Figure 3):

- the three at the bottom, in counterclockwise: T_1, T_2, T_3;

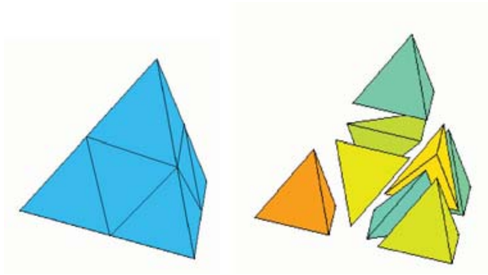

Figure 2: Subdividing a tetraedron by Bey's method.

- the interior ones also in counterclockwise: T_4, T_5, T_6, T_7;
- the one at the top is T_8.

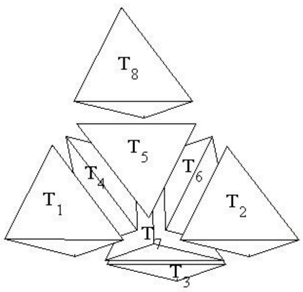

Figure 3: The eight sons of T.

For defining the wavelets, we first need to define the scaling functions. The subdivision tree shown in Figure 4 will help us to index them.

We proceed as follows:

1. Let T be the initial tetrahedron whose volume is $\mathscr{V}(T) = V$.

2. Let $\mathbf{T}^{(1)}$ be the set of the eight sons of T,

$$\mathbf{T}^{(1)} = \{T_1, T_2, ..., T_8\};$$
$$T = \bigcup_{j=1}^{8} T_j; \qquad \mathscr{V}(T_j) = \frac{V}{8}, j = 1, ..., 8.$$

3. For each tetrahedron belonging to $\mathbf{T}^{(1)}$, let us consider its sons. All of them constitute the tetrahedra of level two:

$$\begin{aligned} \mathbf{T}^{(2)} &= \{T_{11}, ..., T_{18}\,; \\ &\quad T_{21}, ..., T_{28}; ....; T_{81}, ..., T_{88}\}\,; \\ \mathscr{V}(T_{ij}) &= \frac{V}{8^2}, i = 1, ..., 8,\ j = 1, ..., 8. \end{aligned}$$

4. Applying the above procedure to the tetrahedra of level two, we obtain the tetrahedra at level three:

$$\begin{aligned} \mathbf{T}^{(3)} &= \{T_{111}, ..., T_{118}; T_{211}, ..., T_{218}; ....; \\ &\quad T_{811}, ..., T_{888}\}\,; \\ \mathscr{V}(T_{ijk}) &= \frac{V}{8^3}, i, j, k = 1, ..., 8. \end{aligned}$$

5. In general, for a tetrahedron T_α in the $n-th$ level, we have:

$$\begin{aligned} T_\alpha &= \bigcup_{j=1}^{8} T_{\alpha,j},\ T_{\alpha,j} \in \mathbf{T}^{(n+1)}; \\ \mathscr{V}(T_\alpha) &= \frac{V}{8^n},\ \mathscr{V}(T_{\alpha,j}) = \frac{V}{8^{n+1}}. \end{aligned}$$

Let us define now the scaling functions. For each tetrahedron, the basic building constructor of the scaling function is the characteristic function of that tetrahedron, multiplied by a constant chosen as to normalize its L^2-norm. So we have:

1. the scaling function for the initial tetrahedron T is: $\varphi = K\chi(T)$, where $\chi(T)$ is the characterictic function of T and K is a constant chosen such that $\|\varphi\|_2 = 1$. Then the associated scaling function for T is:

$$\varphi(T) = \frac{1}{\sqrt{V}}\chi(T).$$

2. for a tetrahedron $T \in \mathbf{T}^{(n)}$, the associated scaling function is:

$$\varphi(T) = \frac{8^{n/2}}{\sqrt{V}}\chi(T).$$

In order to define the Haar wavelets, we need the tree shown in Figure 5. This tree is built using the indices needed for indexing the sons of a tetrahedron. Each wavelet is obtained from an internal node and considering its two sons. More precisely, each wavelet is obtained by normalizing, in L^2-norm, the difference between the characteristic function of the tetrahedra which have index in the first son and the characteristic function of the tetrahedra which have index in the second son. We thus obtain the following

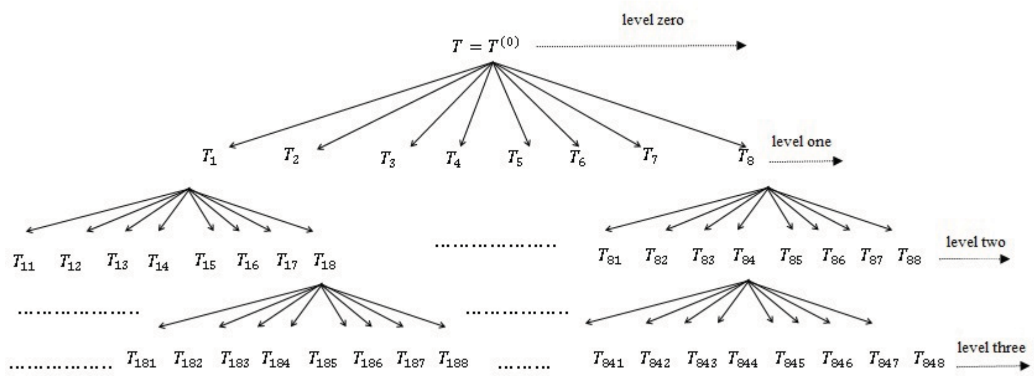

Figure 4: Subdivision tree.

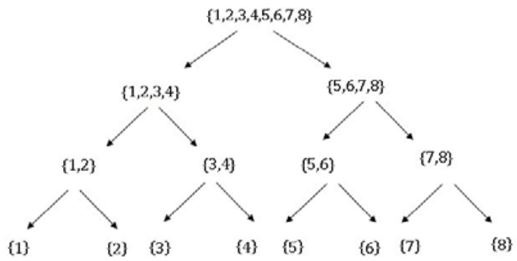

Figure 5: Indices' tree.

seven wavelets:

$$\psi_1 = \frac{8^{n/2}}{\sqrt{V}}[\chi(P_1) - \chi(N_1)],$$
$$P_1 = \cup_{j=1}^{4} T_j, \qquad N_1 = \cup_{j=5}^{8} T_j.$$
$$\psi_2 = \frac{\sqrt{2}\,8^{n/2}}{\sqrt{V}}[\chi(P_2) - \chi(N_2)],$$
$$P_2 = \cup_{j=1}^{2} T_j, \qquad N_2 = \cup_{j=3}^{4} T_j.$$
$$\psi_3 = \frac{\sqrt{2}\,8^{n/2}}{\sqrt{V}}[\chi(P_3) - \chi(N_3)],$$
$$P_3 = \cup_{j=5}^{6} T_j, \qquad N_3 = \cup_{j=7}^{8} T_j.$$
$$\psi_4 = 2\frac{8^{n/2}}{\sqrt{V}}[\chi(P_4) - \chi(N_4)],$$
$$P_4 = T_1, \qquad N_4 = T_2.$$
$$\psi_5 = 2\frac{8^{n/2}}{\sqrt{V}}[\chi(P_5) - \chi(N_5)],$$
$$P_5 = T_3, \qquad N_5 = T_4.$$
$$\psi_6 = 2\frac{8^{n/2}}{\sqrt{V}}[\chi(P_6) - \chi(N_6)],$$
$$P_6 = T_5, \qquad N_6 = T_6.$$
$$\psi_7 = 2\frac{8^{n/2}}{\sqrt{V}}[\chi(P_7) - \chi(N_7)],$$
$$P_7 = T_7, \qquad N_7 = T_8.$$

Finally,

$$\psi_8 = \frac{8^{n/2}}{\sqrt{V}}\chi(T),$$

is the scaling function associated to $T \in \mathbf{T}^{(n)}$.

Proposition: The wavelets defined over a tetrahedron T have the following properties:

1. $\{\psi_j, j = 1, ..., 7\}$ is an orthonormal set.
2. $\int_T \psi_i \, dx^3 = 0, \forall i = 1, ..., 7.$
3. $\{\psi_j, j = 1, ..., 7\}$ is an orthornormal basis for $L^2(T, S, \mu)$, μ being the Lebesgue measure and S the σ-algebra of all tetrahedra generated from Bey's subdivision method of the T tetrahedron.

Properties (i) and (ii) are clear by construction. The demostration of (iii) can be seen in [10]. Since the scaling functions associated to $T \in \mathbf{T}^{(n)}$ and $T_j \in \mathbf{T}^{(n+1)}$ are respectively:

$$\varphi = \frac{8^{n/2}}{\sqrt{V}}\chi(T), \tag{1}$$

and

$$\varphi_j = \frac{8^{\frac{n+1}{2}}}{\sqrt{V}}\chi(T_j), \tag{2}$$

it follows that:

$$\chi(T) = \sum_{j=1}^{8} \chi(T_j) \Rightarrow \frac{8^{n/2}\chi(T)}{\sqrt{V}} = \sum_{j=1}^{8} \frac{8^{n/2}\chi(T_j)}{\sqrt{V}} \Rightarrow$$
$$\varphi = \sum_{j=1}^{8} \frac{1}{\sqrt{8}} \underbrace{\frac{\sqrt{8}\,8^{n/2}\chi(T_j)}{\sqrt{V}}}_{=\varphi_j},$$

so:

$$\varphi = \sum_{j=1}^{8} \frac{1}{\sqrt{8}}\varphi_j. \tag{3}$$

Taking into account (2), the wavelets may be written as follows:

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{8}}\left[\varphi_1+\varphi_2+\varphi_3+\varphi_4-\right.\\
&\quad \left.-\left(\varphi_5+\varphi_6+\varphi_7+\varphi_8\right)\right],\\
\psi_2 &= \frac{1}{2}\left[\varphi_1+\varphi_2-\left(\varphi_3+\varphi_4\right)\right],\\
\psi_3 &= \frac{1}{2}\left[\varphi_5+\varphi_6-\left(\varphi_7+\varphi_8\right)\right],\\
\psi_4 &= \frac{1}{\sqrt{2}}\left[\varphi_1-\varphi_2\right],\\
\psi_5 &= \frac{1}{\sqrt{2}}\left[\varphi_3-\varphi_4\right],\\
\psi_6 &= \frac{1}{\sqrt{2}}\left[\varphi_5-\varphi_6\right],\\
\psi_7 &= \frac{1}{\sqrt{2}}\left[\varphi_7-\varphi_8\right]. \qquad (4)
\end{aligned}$$

It must be noted that at level $(n+1)$ there are 7×8^n wavelets. For example at level two there are 7×8^1 wavelets.

3.1 Multiresolution analysis spanned by the Haar-like wavelets over a tetrahedron

Wavelets are closely related to the concept of multiresolution analysis [1, 2, 3]. In [10] it is shown that orthogonal Haar-like wavelets for general measures fit into this concept. Since our wavelets are a particular case of those ones, they also fit into this concept. In order to write the multiresolution analysis let us adopt the following notation:

- T_α a parent tetrahedra and T_{β_i}, $i=1,..,8$ its sons accordingly to Bey's subdivision method.
- φ_α: scaling function associated to T_α
- φ_{β_i}: scaling function associated to T_{β_i}

With this notation equation (3) and equations (4) are written as follows:

$$\varphi_\alpha = \sum_{i=1}^{8}\frac{1}{\sqrt{8}}\varphi_{\beta_i} \qquad (5)$$

and:

$$\psi_j = \sum_{i=1}^{8} g_{\beta_{i,j}}\varphi_{\beta_i},\ j=1,...,7, \qquad (6)$$

where the coefficients $g_{\beta_{i,j}}$ are shown in Table 1.

Table 1: Wavelets' coefficients

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
$j=1$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$\frac{-1}{\sqrt{8}}$	$\frac{-1}{\sqrt{8}}$	$\frac{-1}{\sqrt{8}}$	$\frac{-1}{\sqrt{8}}$
$j=2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	0	0	0	0
$j=3$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$
$j=4$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0	0	0	0	0	0
$j=5$	0	0	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0	0	0	0
$j=6$	0	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	0	0
$j=7$	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

Now we define the multiresolution spaces. Let us note:

1. $span(A)$: subspace generated by the elements of A, A being a set of functions.
2. $\langle f,g\rangle = \int_T fg dx^3$, $f,g\in L^2(T)$.
3. $\boldsymbol{\varphi^n} = \{\varphi^n : \text{scaling functions at level } n\}$.
4. $\boldsymbol{\psi^n} = \left\{\psi_j^n : \psi^n : \text{ wavelets at level } n\right\}$.

For the $(n+1)-th$ level in the subdivision tree of Figure 4, we define the scaling spaces V^{n+1} and the detail spaces W^{n+1} as follows:

$$V^{n+1} = span(\boldsymbol{\varphi}^{n+1}) \text{ and } W^{n+1} = span(\boldsymbol{\psi}^{n+1})$$

If the subdivision tree has two levels, $(n+1)$ y n, we have:

$$V^{n+1} = V^n \oplus W^n,$$

so a function $f\in V^{n+1}$ has the two following representations:

$$f = \sum_{\varphi_\beta\in\boldsymbol{\varphi}^{n+1}} a_\beta^{(n+1)}\varphi_\beta,$$

being $a_{\beta'}^{(n+1)} = \langle f,\varphi_{\beta'}\rangle$, and:

$$f = \sum_{\varphi_\alpha\in\boldsymbol{\varphi}^n} a_\alpha^n\varphi_\alpha + \sum_{\psi_\gamma\in\boldsymbol{\psi}^n} c_\gamma^n\psi_\gamma,$$

being $a_{\alpha'}^n = \langle f,\varphi_{\alpha'}\rangle$ and $c_{\gamma'}^n = \langle f,\psi_{\gamma'}\rangle$.

As always the operations needed for passing from one representation to the other are found using the fast wavelet transform which consists of two steps: analysis and synthesis.

Analysis

Given the coefficients $a_\beta^{(n+1)}$ at level $(n+1)$, the coefficients a_α^n and c_γ^n at level n, can be obtained using the direct wavelet transform:

$$\begin{aligned}
a_\alpha^n &= \frac{1}{\sqrt{8}}\sum_\beta a_\beta^{(n+1)}\\
c_\gamma^n &= \sum_\beta g_{\gamma,\beta}a_\beta^{(n+1)},
\end{aligned}$$

where $g_{\gamma,\beta}$ are the coefficients in Table 1. Using those values, c_γ^n can be written:

$$
\begin{aligned}
c_{\gamma_1}^n &= \frac{1}{\sqrt{8}}\left(a_{\beta_1}^{(n+1)}+a_{\beta_2}^{(n+1)}+a_{\beta_3}^{(n+1)}+a_{\beta_4}^{(n+1)}\right)-\\
&\quad -\frac{1}{\sqrt{8}}\left(a_{\beta_5}^{(n+1)}+a_{\beta_6}^{(n+1)}+a_{\beta_7}^{(n+1)}+a_{\beta_8}^{(n+1)}\right)\\
c_{\gamma_2}^n &= \frac{1}{2}\left(a_{\beta_1}^{(n+1)}+a_{\beta_2}^{(n+1)}\right)-\frac{1}{2}\left(a_{\beta_3}^{(n+1)}+a_{\beta_4}^{(n+1)}\right)\\
c_{\gamma_3}^n &= \frac{1}{2}\left(a_{\beta_5}^{(n+1)}+a_{\beta_6}^{(n+1)}\right)-\frac{1}{2}\left(a_{\beta_7}^{(n+1)}+a_{\beta_8}^{(n+1)}\right)\\
c_{\gamma_4}^n &= \frac{1}{\sqrt{2}}\left(a_{\beta_1}^{(n+1)}-a_{\beta_2}^{(n+1)}\right)\\
c_{\gamma_5}^n &= \frac{1}{\sqrt{2}}\left(a_{\beta_3}^{(n+1)}-a_{\beta_4}^{(n+1)}\right)\\
c_{\gamma_6}^n &= \frac{1}{\sqrt{2}}\left(a_{\beta_5}^{(n+1)}-a_{\beta_6}^{(n+1)}\right)\\
c_{\gamma_7}^n &= \frac{1}{\sqrt{2}}\left(a_{\beta_7}^{(n+1)}-a_{\beta_8}^{(n+1)}\right)
\end{aligned}
$$

Synthesis
Given the coefficients a_α^n and c_γ^n at level n, the coefficients $a_\beta^{(n+1)}$ are obtained as follows:

$$a_\beta^{(n+1)} = \frac{1}{\sqrt{8}}a_\alpha^n + \sum_\gamma c_\gamma^n g_{\gamma,\beta} \tag{7}$$

4 Example

The Haar wavelets defined in this work are appropriate to represent attributes (e.g. color, density) of a volume that can be represented with a finite union of tetrahedra. These attributes can be thought of as scalar functions with domain on the tetrahedra. For each tetrahedron, the attribute is generally non-uniform but every tetrahedron can be subdivided until the attribute is constant on each one of the smallest tetrahedra. This leads to a representation of the attribute on the tetrahedron using piecewise constant functions and the Haar basis defined above is a base for this kind of functions defined on tetrahedra.
In this example, we choose as scalar function the density function associated to the interior and the surface of a given tetrahedron. We consider that the tetrahedron has a non-uniform density and then it is subdivided using Bey's method until the density on each one of the smallest tetrahedra is uniform, *i.e.* can be represented with a piecewise constant function. In this way we have a tetrahedron represented as a union of many tetrahedra as necessary and this is the tetrahedral representation of the given tetrahedron at its finest resolution. The values for the density function defined on the finest resolution were randomly generated and can be seen in Figure 6. The coefficients at coarser resolutions were found using the fast wavelet transform and, in order to give a visual representation of the values of those coefficients, we chose to map on

Initial random values, finest resolution (*n*=2)	Scaling and wavelet coefficients, first step of the multiresolution analysis (*n*=1)	Scaling and wavelet coefficients, second step of the mutiresolution analysis (n=0)
0,6385	**1,9946**	**4,8201**
0,5313	0,0969	-0,2889
0,9436	-0,3091	-0,1566
0,8444	0,019	0,0824
0,7847	0,0758	0,6658
0,5762	0,0701	-0,0117
0,7093	0,1474	-0,4654
0,6135	0,0677	-0,2203
0,0336	**1,0530**	
0,1056	0,0483	
0,6377	-0,6395	
0,7805	0,0088	
0,0358	-0,0509	
0,6834	-0,101	
0,1194	-0,4579	
0,5822	-0,3272	
0,0688	**1,6720**	
0,6110	-0,0366	
0,9577	-0,4766	
0,6753	0,0602	
0,7218	-0,3834	
0,5466	0,1997	
0,6073	0,1239	
0,5407	0,0471	
0,5309	**1,6886**	
0,7788	-0,2792	
0,6761	0,3135	
0,0067	-0,14	
0,6074	-0,1753	
0,6444	0,4733	
0,6619	-0,0262	
0,8699	-0,1471	
0,6544	**1,5184**	
0,0908	-0,0906	
0,6718	-0,2644	
0,6022	0,2483	
0,7384	0,3985	
0,6476	0,0492	
0,7703	0,0642	
0,1192	0,4604	
0,8200	**2,1766**	
0,5271	-0,0849	
0,6951	-0,132	
0,9160	-0,0027	
0,9174	0,2071	
0,6790	-0,1562	
0,6620	0,1686	
0,9398	-0,1964	
0,7184	**1,6093**	
0,8754	-0,4334	
0,0680	0,7623	
0,0012	-0,0431	
0,7655	-0,111	
0,6358	0,0472	
0,8419	0,0917	
0,6456	0,1388	
0,9686	**1,9209**	
0,5181	0,1471	
0,6678	0,0243	
0,7702	-0,218	
0,0911	0,3186	
0,9452	-0,0724	
0,8329	-0,6039	
0,6393	0,1369	

Figure 6: Two steps in the multiresolution decomposition: coefficients

to grey scale the coefficients of the scaling functions and on to colour scale the wavelets coefficients. So there are seven different colours, one for each wavelet. In both scales, the tetrahedra have been coloured depending on the absolute value of coefficients. When the coefficient value is zero, the corresponding tetrahedron is white. If the absolute value of a coefficient increases, then it also increases the saturation of its corresponding colour. In Figure 7, two steps can be

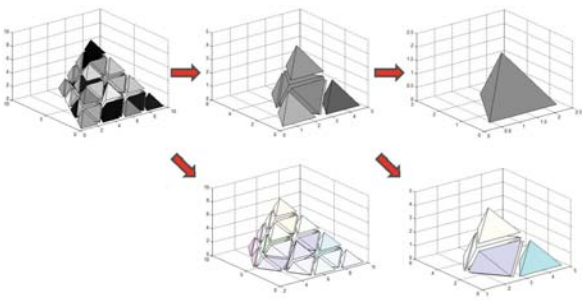

Figure 7: Two steps in the multiresolution decomposition of the density function.

seen in the multiresolution decomposition of the density function defined on a tetrahedron. On the left, the density function defined over space V_2. The first step is represented in the middle column. The grey tetrahedra represent the density function on the space V_1; below, the details on the space W_1. In the second step (third column), the grey tetrahedra also represent the density function but this time on the space V_0 and below, the details on the space W_0. Finally, Figure 8 is a

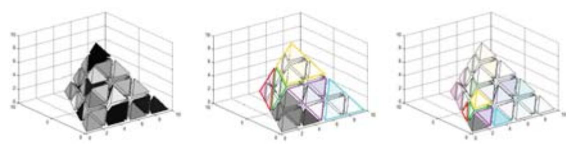

Figure 8: Compact representation.

compact representation of the decomposition that was shown in Figure 7. This representation is analogue to the one used for images.

5 Conclusions

Many applications such as those involving Internet 3D models, collaborative CAD, interactive visualization and multiplayer video games cope with the problem of representing functions, at different level of detail, over a volume. Typically this volume is represented by a regular 3D grid of tetrahedra (which we supposed given) and the issue is to represent a function (color, brightness, density) defined over the object. As wavelets have been proved to be a powerful tool for representing general functions and large data sets accurately, we look for bases in the field of wavelets and particularly wavelets over tetrahedra because they are the basic blocks building for a volume. Following this idea, in this work we have constructed a wavelet basis over a tetrahedron which enables the multiresolution representation of functions defined over a tetrahedron and, consequently, the multiresolution representation of functions defi ned over a tetrahedrilized volume. We have supposed that the attributes are regularly distributed on the grid and because of this we have used nested tetrahedral grids generated by a recursive method (Bey's method) that are suitable to deal with that kind of data. Using the defined basis it is possible to represent any function defined on a volume (surface and interior) using the multiresolution approach. Our actual work is focused on defining a wavelet basis on non nested tetrahedral meshes which are desirable to deal with irregular distributed data points.

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Competing interests

The authors have declared that no competing interests exist.

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